Asymptotic Path Curves

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Asymptotic Curves

The results described below were first published more concisely in Ref. 2.

A curve *C* in a surface *S* passing through a point P in *S* possesses curvature at that point. In the tangent plane to *S* at P a flat pencil of tangents to the surface exists, and a curve in the surface passing though P will have one of the lines of the pencil as its tangent at P. The curvature of all curves at P touching the set of tangents at P may be positive, zero or negative depending upon the nature of the surface. If a surface has points where the curvature of *C* may be positive or negative for different tangents then it is said to have negative Gaussian curvature. There will be two tangents where *C* has zero curvature, and the directions of those tangents are called *asymptotic directions*. An asymptotic line (or curve) in a surface is such that all its tangents are asymptotic directions. Generally two asymptotic curves pass through every point of a surface which has negative Gaussian curvature. A ruled hyperboloid is an example of a surface with negative curvature, as will be explained below, while a sphere or ellipsoid has positive curvature and therefore no asymptotic curves.

Thus an asymptotic curve is a kind of boundary between positive and negative curvature on a surface. For example, consider a ruled hyperboloid:

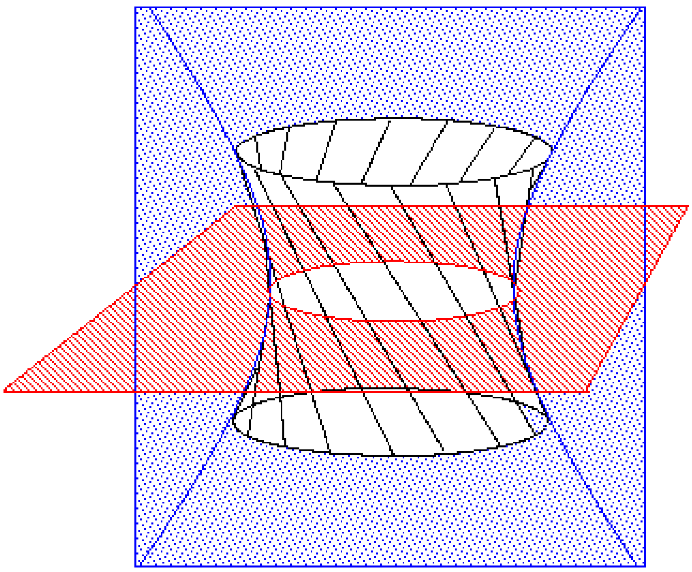

Figure 1

The red plane intersects it in a circle, a curve which has positive curvature, while the blue plane intersects it in a hyperbola, which has negative curvature. If we rotate the plane from red to blue, at one position it meets the surface in two straight lines called *rulers*, which have zero (or no) curvature. Those lines are asymptotic curves because they mark the transition between cross sections with positive and negative curvature. There are many asymptotic curves on a surface, and the rulers are the asymptotic curves (lines) in this case. It is clear that no such argument can be applied to an ellipsoid as all intersecting planes meet it in ellipses, which have positive curvature. Another way of expressing all this is to say that curves with positive curvature have their *centres of curvature* inside the surface, while those with negative curvature have their centres of curvature outside. The asymptotic curves are a transition between these two cases. For a circle the centre of curvature is obviously its centre, while for other curves it varies and at a given point it is the centre of the tangential circle in the *osculating plane* which has the same curvature as the curve at that point.

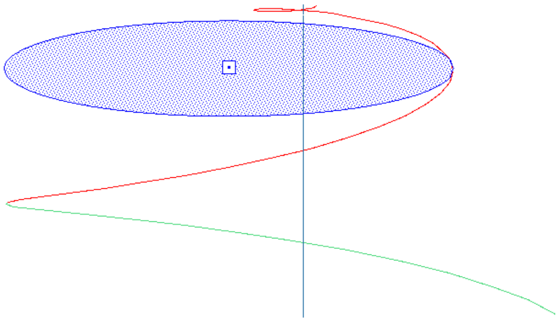

Figure 2

The curvature is the reciprocal of the radius of the tangential circle.

An osculating plane at a point P on a curve is that plane in which the tangent at P is momentarily turning i.e. the curve momentarily lies in that plane. In the diagram below the red plane represents a tangent plane and the three tangents illustrate what is meant, although they should of course be 'consecutive' tangents as the curve only lies in the plane at the indicated point of tangency:

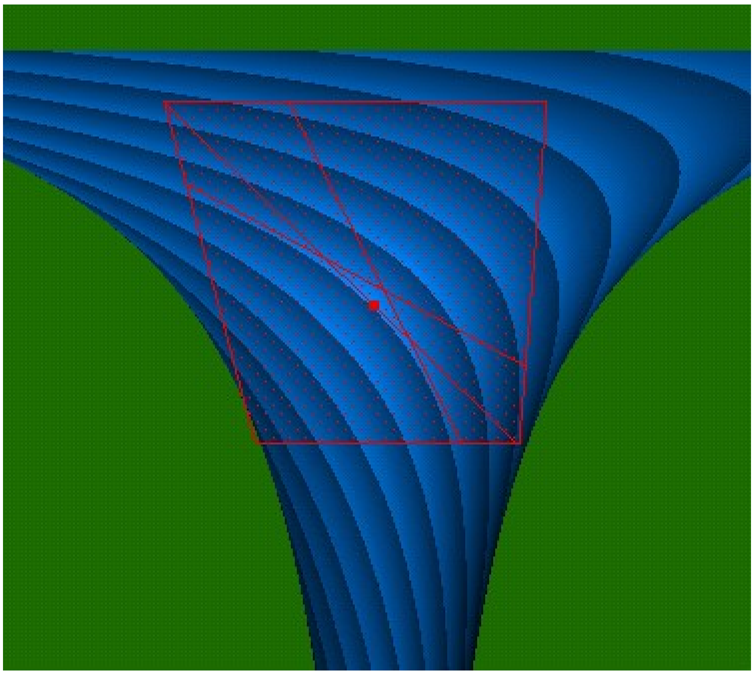

Figure 3

For more complex surfaces than hyperboloids there may exist points such that all the curves through them have their centres of curvature on only one side of the surface, known as *elliptical points*, and *hyperbolic points* with centres of curvature on both sides for the various curves passing through it. A surface must possess hyperbolic points for it to contain asymptotic lines. The spiralling curves on the vortex above are its asymptotic curves, as we will see later.

Since the curvature of a curve tangential to an asymptotic direction at a point is zero at that point, it is neither turning towards one side of the surface nor towards the other, and hence demarcates those curves with their centres of curvature on one side of the surface from those with them on the other. The tangent to the curve is thus momentarily turning *in the surface* so that the osculating plane coincides with the tangent plane to the surface. Thus the asymptotic curves are those curves whose osculating planes coincide with the tangent planes to the surface at each point of the curve.

Path Curve Surfaces

By *path curve surfaces* we mean surfaces formed by a complete covering of path curves. See References 1 and 3 for an explanation of path curves. One of the simplest such surfaces is a circularly symmetrical vortex such as that illustrated in Figure 3. As shown in those references it is generated by two logarithmic spirals in two parallel planes, as in Figure 4 below.

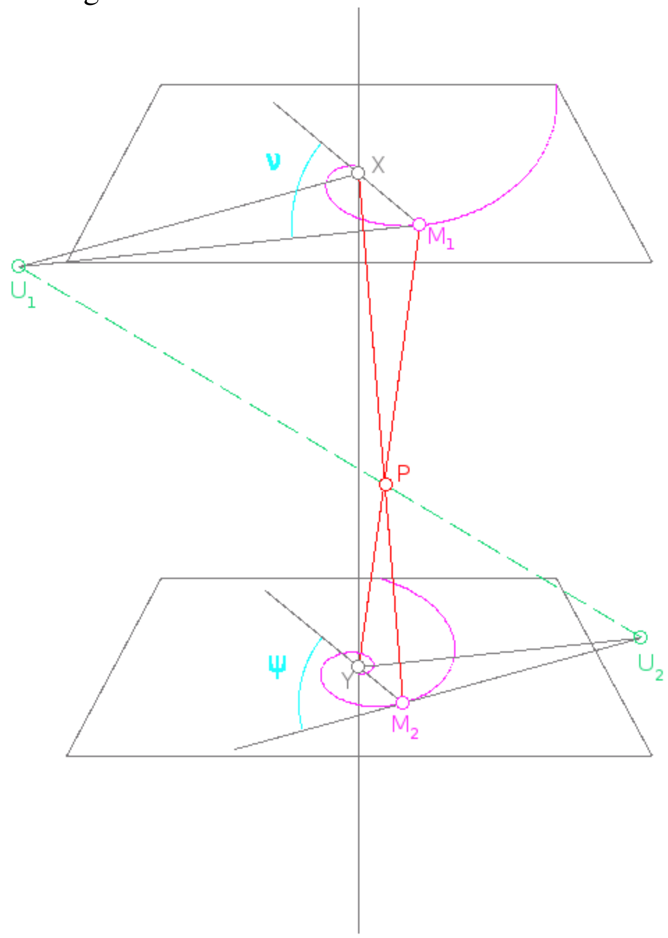

Figure 4

The path curve arises by rotating a plane about XY, which meets the spiral in the top invariant plane in M_1 and that in the bottom plane in M_2. Then XM_2 meets YM_1 in a point P of the path curve. XU_1 is drawn parallel to the tangent M_2U_2 in the bottom plane, and YU_2 is drawn parallel to M_1U_1 in the top plane. Then U_1 is the intersection of XU_1 and M_1U_1, and similarly for U_2.

The osculating plane of a path curve is illustrated below:

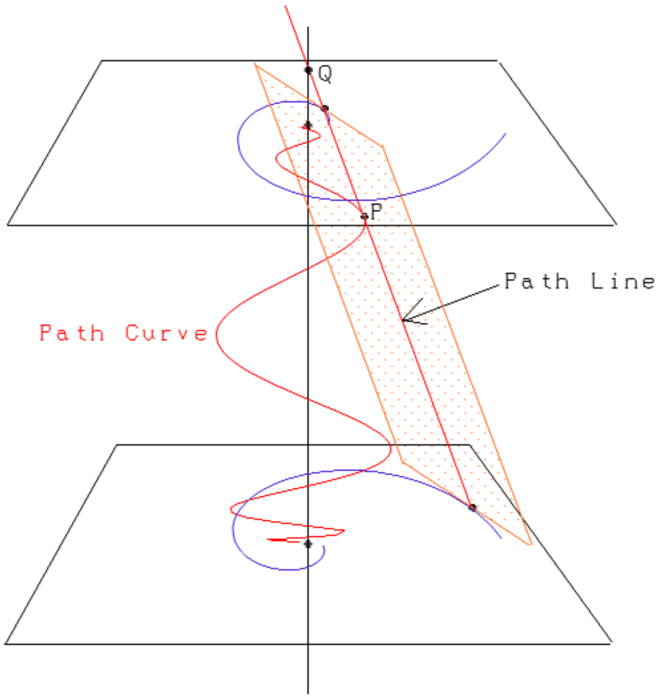

Figure 5

The osculating plane is shown shaded red, being tangential to the top and bottom spirals. The line joining the points of contact is called the path line which passes through the point of contact P on the path curve (shown as an egg in this diagram). The red plane is momentarily turning about the path line as it remains tangential to the spirals i.e. the path line is momentarily fixed, and thus P is momentarily moving along that line. Thus P is moving in the red plane which is the osculating plane.

The construction in Figure 4 makes the tangents to the spirals through U_1 and U_2 (not shown) parallel to each other, so that U_1U_2 is the path line along which P is moving and about which the osculating plane is turning. The following diagram clarifies this, where the fact that the angle between a radius and tangent of a logarithmic spiral is constant e.g. the angles ν and ψ in Figure 4.

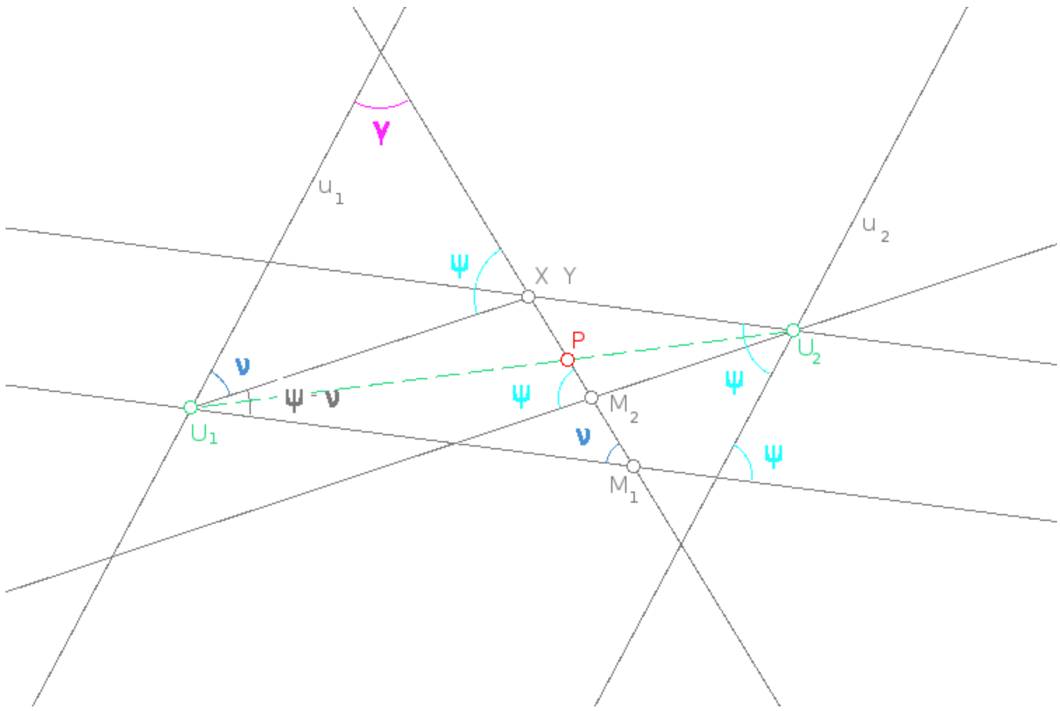

Figure 6

This shows the top and bottom invariant planes of Figure 4 superimposed. X and Y coincide and M_1 and M_2 lie on the line in which the plane M_1M_2P intersects the figure. The tangent to the top spiral is M_1U_1 at an angle ν to M_1M_2, and similarly M_2U_2 is at an angle ψ to M_1M_2. Then XU_1 is parallel to the tangent M_2U_2 and YU_2 is parallel to M_1U_1. The angle XU_1M_1 is then ψ-ν as shown. The spiral in the top plane passing through U_1 has its tangent u_1 at U_1 at an angle ν to XU_1 as shown, so that the angle between u_1 and M_1U_1 equals ψ. Similarly the tangent u_2 through U_2 to the spiral in the bottom plane is at an angle ψ to YU_2 so that the angle between u_2 and M_1U_1 is ψ. Thus u_1 and u_2 are parallel lines where the osculating plane touching the spirals at U_1 and U_2 intersects the top and bottom planes. The line U_1U_2 is thus the path line through P where it meets M_1M_2.

For the osculating plane also to be a tangent plane, the tangents u_1 and u_2 must lie at an angle of 90° to the plane XYP i.e. the angle γ shown between u_1 and M_1M_2 must be 90° which means

$$\nu+\psi=90^{\circ} \qquad (1)$$

This is the condition for the path curve to be an asymptotic curve in the case of horizontal circular symmetry such as for the vortex in Figure 3.

However more general surfaces arise when the horizontal cross-sections of the surface are logarithmic spirals, as illustrated below:

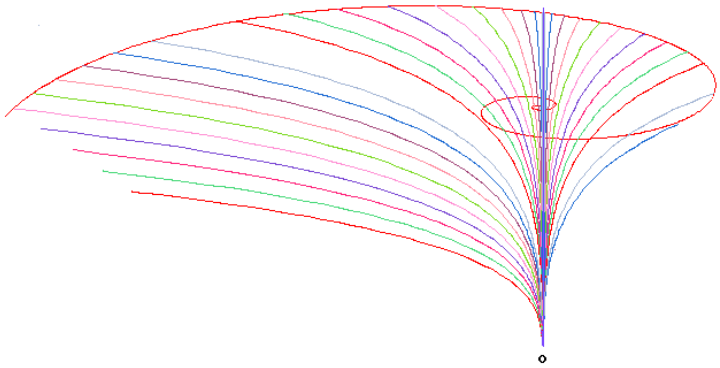

Figure 7

Here vertical path curves that are vortical all with the same λ-value pass through a horizontal logarithmic spiral to generate the surface. In the equation $r = r_0 e^{\beta\theta}$ for the spirals, β=cotϕ where ϕ is the angle between a radius and tangent of the spiral. For circles ϕ=90° which was assumed when using γ=90° in (1) above for circular symmetry. More generally γ=180-ν-ψ in Figure 6, and γ= ϕ for the logarithmic spirals, so the more general condition for an asymptotic curve is

$$180\text{-}\nu\text{-}\psi = \phi \tag{2}$$

In Ref. 1 the parameter α is defined by

$$\lambda = \frac{\epsilon + \alpha}{\epsilon - \alpha} \tag{3}$$

where λ and ϵ define the path curve, and furthermore it is shown there that cotν=ϵ+α in the top invariant plane and cotψ= ϵ-α in the bottom plane. However that was for eggs rather than vortices, so for a vortex we must have cotψ=α-ϵ as the spiral winds in the opposite sense from that for an egg. It follows that

$$\begin{aligned}
2\epsilon &= \cot\nu - \cot\psi \\
2\alpha &= \cot\nu + \cot\psi \\
\cot\psi &= \cot(180 - \nu - \phi) = -\cot(\nu + \phi) \\
&= \frac{1 - \cot\nu\cot\phi}{\cot\nu + \cot\phi}
\end{aligned}$$

and then

$$\lambda = \frac{\epsilon + \alpha}{\epsilon - \alpha} = -\frac{\cot\nu}{\cot\psi} = \frac{\cot\nu + \cot\phi}{\cot\phi - \tan\nu}$$

This implies a quadratic equation in cotν the solution of which is

$$\cot\nu = k \pm \sqrt{k^2 - \lambda}$$
where
$$k = \frac{(\lambda - 1)\cot\phi}{2} \tag{4}$$

Hence given ϕ and λ we may calculate ν, ϵ and α.

From (2)

$$\begin{aligned}\cot\phi &= -\cot(\nu+\psi)\\ &= \frac{1-\cot\nu\cot\psi}{\cot\nu+\cot\psi}\\ &= \frac{1+(\epsilon+\alpha)(\epsilon-\alpha)}{\epsilon+\alpha-\epsilon+\alpha}\end{aligned}$$

i.e.

$$\cot\phi = \frac{1}{2}\left[\frac{1+\epsilon^2}{\alpha}-\alpha\right] \tag{5}$$

This equation was first derived by Dr. Georg Unger using differential geometry techniques.

We will now derive the parameters of the surface for which the path curve is an asymptotic curve. That surface is defined by the parameter β=cotϕ for the spiral cross-sections, and μ which equals the λ-value of the vertical cross sections c.f. Figure 7. Note that the surface is fully determined by β and μ. μ is defined in Ref. 1 Appendix 2 by

$$\mu = \frac{\epsilon+\alpha+\beta}{\epsilon-\alpha-\beta} \tag{6}$$

Substituting for α in (5) using (3) and simplifying gives

$$\beta = \frac{4\lambda\epsilon^2+(1+\lambda)^2}{2\epsilon(\lambda^2-1)} \tag{7}$$

Then substituting for α in (6) again using (3), and for β using (7), we find

$$\mu = -\frac{4\epsilon^2\lambda^2+(1+\lambda)^2}{4\epsilon^2+(1+\lambda)^2} \tag{8}$$

This solves the problem of finding the surface for which particular path curves are its asymptotic curves, and is unique for each λ ϵ pair. Note that μ is necessarily negative which means the vertical cross-sections are vortex profiles, and that the Gaussian curvature is negative as required for asymptotic curves to exist. It is perhaps surprising that λ may be positive i.e. the asymptotic curves may be egg path curves. The following diagram shows an example of a surface defined by the spiral cross-sections showing one of each of the asymptotic path curves:

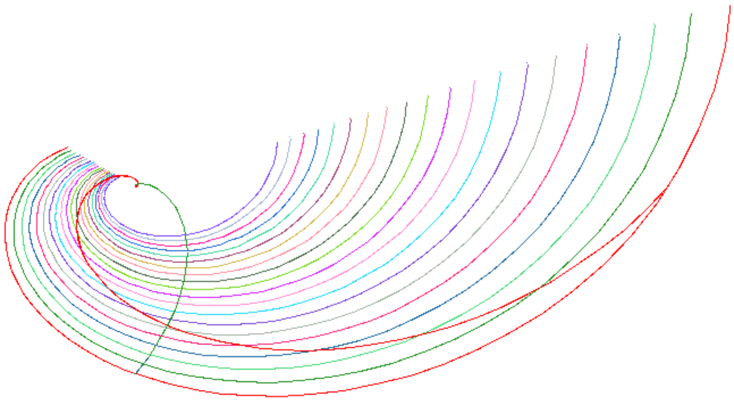

Figure 8

If instead we know β and μ for a surface then we can solve (7) and (8) to find the parameters of its asymptotic path curves:

$$\epsilon^2 = -\frac{(1+\beta^2)(1+\mu)^2}{4\mu} \tag{9}$$

$$\lambda = \frac{\mu \pm \beta(1+\mu)\sqrt{-\mu(1+\beta^2)}}{1+\beta^2(1+\mu)} \tag{10}$$

μ must be negative for real ϵ and (10) gives the two λ-values of the asymptotic curves which have the same absolute value of ϵ.

Another interesting case is to find the surface with asymptotic curves having two given values of λ. If these are λ_1 and λ_2 then we may use (10) twice for these two values and eliminate μ to give

$$\beta^2 = \frac{-(\lambda_1-\lambda_2)^2}{2(\lambda_1+\lambda_2)(\lambda_1+1)(\lambda_2+1)} \tag{11}$$

The denominator must be negative for real β so that either

$\lambda_1 < -1$ and $\lambda_2 < -1$

or $-\lambda_1 < \lambda_2 < -1$

or $-1 < \lambda_2 < -\lambda_1$

or $1 < -\lambda_1 < \lambda_2$

The two λs must be chosen accordingly, at least one being negative. Solving likewise for μ gives

$$\mu = \frac{2\lambda_1\lambda_2+\lambda_1+\lambda_2}{\lambda_1+\lambda_2+2} \tag{12}$$

Thus for some pairs of values of λ_1 and λ_2 it is possible to find a surface for which they define the asymptotic curves. Having found β and μ, ϵ is found from (9).

References

1. Edwards, Lawrence, *The Vortex of Life* Floris Press, Edinburgh 1993.
2. Thomas N.C., *Asymptotic Lines & the Pivot Transform*,
Mathematisch-Physikalische Korrespondenz Nr. 161, 24/6/1991,
published by the Mathematical-Astronomical Section, Dornach, Switzerland.
3. Thomas N.C., *Path curves* (available at http://healingwaterinstitute.org, *Mathematics* tab).